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A SIMPLE DISCUSSION OF LOGARITHMIC ERRORS.

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PROBLEM VI.

To determine whether it is more accurate to use the apparent *or the* true *tabular difference.*

Case I. A number is given and the corresponding mantissa required. In Problem I it has been proven that if the apparent tabular difference be employed the resulting error is $xc' + (1 - x)c$. If now, the true tabular difference be, used, the error of the interpolated mantissa will be (using the nomenclature of Problem II)

$$t + c + x(t' - t) + x(c' - c) - [t + x(t' - t) + x(c' - c)],$$

which reduces to c, as it evidently should. The probable error of c is 0.25 and, if the interpolated mantissa be restricted to n places, another error whose probable value is 0.25, is introduced, so that the final error is $0.25\sqrt{2}$ or 0.35, which is larger than the probable error of the ordinary method, in which the apparent tabular difference is used (see Problem I).

Case II. A mantissa is given and the corresponding natural number is required. If the true tabular difference be used, we divide d by $D + c' - c$ and get

$$\frac{d}{D} - \frac{d}{D^2}c' + \frac{d}{D^2}c + \frac{d}{D^3}(c' - c) + \ldots.$$

Comparing this with the first quotient in Problem II, neglecting terms which contain powers of D higher than the second, and multiples of c and c', we find the error to be $\frac{c}{D}$. Now, the corresponding error of the usual method is found in Problem II to be

$$\frac{d}{D^2}c' + \left(\frac{c}{D} - \frac{d}{D^2}c\right),$$

whose probable value has been shown to be $\frac{0.198}{D}$. But the probable value of $\frac{c}{D}$ is $\frac{0.25}{D}$. We conclude then that in neither case is it advisable to use the true tabular difference.

PROBLEM VII.

To find the probable error of an n-place interpolated mantissa when the corresponding n-place natural number has a probable error of ε in the n^{th} place.

It is plain that we now introduce a new error of $\frac{\varepsilon D}{10^f}$, where f denotes the difference between the number of figures in the tabular argument and the number in the tabular function. Assuming $8(10)^{f-1}$ as the probable value of D and taking 0.31 from Problem I as the average of the probable errors due to other causes, we get for the final result

$$\text{Probable error} = \sqrt{[0.64\varepsilon^2 + (0.31)^2]}.$$

But if both the magnitude and sign of ε are known, and a particular value of D be given, the actual error due to this cause can be quickly computed.

PROBLEM VIII.

To find the probable error of an n-place interpolated natural number when the corresponding mantissa has a probable error of ε in the n^{th} place.

This error of ε will evidently introduce an error of $\frac{\varepsilon}{D}10^f$, and the total error is seen by Problem II to be

$$\sqrt{\left[\left(\frac{\varepsilon}{D}10^f\right)^2 + \left(\frac{0.198}{D}10^f\right)^2 + (0.25)^2\right]}.$$

If we assume that $8(10)^{f-1}$ is the probable value of D, this expression becomes

$$\sqrt{[1.56\varepsilon^2 + (0.35)^2]}.$$

If the value of D be 10, the effect of the error ε is the same in Problem VII as in Problem VIII; if $D < 10$, the effect is greater in VIII than in VII; and, if $D > 10$, the effect is greater in VII tnan in VIII.

PROBLEM IX.

Given an n-place table of logarithms of natural numbers, in which the last figure of each mantissa, if too large, is marked with a minus sign: to find the probable error of the mantissa corresponding to a given n-place number, if the mantissa be carried to $n + 2$, $n + 1$, and n-places respectively.

For simplicity, consider a 5-place table, and apply to the tabular mantissas, just before and after the mantissa to be found, the correction + 0.25 or — 0.25, according as their last figures are unmarked or are marked with a minus sign. Thus each tabular mantissa is carried to seven places and ends with the figure 5. Use the difference between these as the tabular difference, and carry the interpolated mantissa to seven places. Now, the probable value of c or c' is 12.5 units of the seventh place; reasoning as in Problem I, and remembering that no figures are rejected, we find that the probable error of the interpolated mantissa, if $x = 0.1$ is $\sqrt{[(0.1)^2 + (0.9)^2]}\,.\,(12.5)$. We then form the following table: —

x	Error.	x	Error.
0.0	12.5	0.5	$(12.5)\,.\,\sqrt{(.50)}$
0.1	$(12.5)\,.\,\sqrt{(.82)}$	0.6	$(12.5)\,.\,\sqrt{(.52)}$
0.2	$(12.5)\,.\,\sqrt{(.68)}$	0.7	$(12.5)\,.\,\sqrt{(.58)}$
0.3	$(12.5)\,.\,\sqrt{(.58)}$	0.8	$(12.5)\,.\,\sqrt{(.68)}$
0.4	$(12.5)\,.\,\sqrt{(.52)}$	0.9	$(12.5)\,.\,\sqrt{(.82)}$

The average of these errors is $(0.813)\,.\,(12.5)$, or 10.2 units of the seventh place.

Let us find the probable error if the mantissa be restricted to 6 places. Half the time the tabular differences do not extend beyond the 5th place and half the time they end in 5 of the sixth place. When they do not extend beyond the 5th place and are multiplied by values of x ranging from 0.1 to 0.9, the last figure of the proportional part found affects the 6th place only, and the interpolated mantissa, when carried to seven places, ends in the figure 5. But if the tabular difference end in 5 of the 6th place, the proportional part ends four-ninths of the time in 0 and five-ninths of the time in 5 of the seventh place. So the 7-place interpolated mantissa ends in 0 five-ninths of the time and in 5 four-ninths of the time. Hence in the long run, in 13 cases out of 18, the mantissas end in 5 of the 7th-place and in 5 cases out of 18 they end in 0. Including the case where $x = 0$, so that no interpolation is made, we see that for fifteen-twentieths or three-fourths of the time the resulting mantissa ends in 5 and for one-fourth of the time in 0.

If now the interpolated mantissas be restricted to 6 places, we find that for three-fourths of the time the probable error expressed in units of the 7th-place is $\sqrt{[(10.2)^2 + 5^2]}$, which equals 11.4; for one-fourth of the time the error is 10.2. Combining these, having regard to their weights, we obtain 11.1 as the average of the probable errors. I have taken a series of 90 uniformly distributed numbers and restricted the corresponding mantissa to 6 places. The mean of the resulting errors was 12.1 in the seventh place. Finally, if the mantissa be restricted to 5 places, the probable value of the rejected figures will be 25 units of the seventh place, and since the probable error of the mantissas, when carried to 7 places, is 10.2 units, we have for the present case

$$\begin{aligned}\text{Probable error} &= \sqrt{[25^2 + (10.2)^2]}\\ &= 27.0 \text{ units of the seventh place}\\ &= 0.27 \text{ units of the fifth place.}\end{aligned}$$

In the case of the 90 mantissas just mentioned, the average of the errors was

0.25. Comparing the above result with that of Problem I we see that when the interpolated mantissas are restricted to 5 places, the extra labor involved in the use of the table just discussed receives no adequate compensation in increase of accuracy.

It is evident that the maximum error, when the mantissas are carried to 7 places, is 25 units of seventh place; when carried to 6 places, it is 30 units; and when restricted to 5 places, it is 75, or 0.75 of a unit of the fifth place.

PROBLEM X.

Given an n-place mantissa, and the logarithm-table mentioned in Problem IX: to find the probable error of the corresponding number if carried to n + 1 *and to n places respectively.*

We use the notation of Problem II and discuss a 5-place table, remarking that t and t' now denote two 7-place mantissas which end in 25 or 75 and that the probable value of c or c' is 0.125 in units of the fifth place. If we carry the result to 6 places, the formula for the probable resulting error is in units of the sixth place

$$\text{Error} = \sqrt{\left(\left[10\frac{1.25}{D}\sqrt{\left(\frac{d}{D}\right)^2 + \left(1-\frac{d}{D}\right)^2}\right]^2 + (0.25)^2\right)}.$$

Then, reasoning as in Problem II, the average of the probable errors becomes

$$\sqrt{\left[\left(\frac{9.9}{D}\right)^2 + (0.25)^2\right]}.$$

If the numbers corresponding to the given mantissas be uniformly distributed throughout the table, the average of all the values of the term $\frac{9.9}{D}$ will be $\frac{9.9}{8}$ or 1.24. Substituting this value in the expression just given, we find the result to be 1.26. But if the mantissas be uniformly distributed the average of the values of $\frac{9.9}{D}$ is 0.96, and the average of the probable errors is 0.99.

When the resulting natural numbers are restricted to 5 places, the value of the probable error in units of the fifth place is

$$\sqrt{\left(\left[\frac{1.25}{D}\sqrt{\left(\frac{d}{D}\right)^2 + \left(1-\frac{d}{D}\right)^2}\right]^2 + (0.25)^2\right)}.$$

Then the average of the probable errors is

$$\sqrt{\left[\left(\frac{0.99}{D}\right)^2 + (0.25)^2\right]}.$$

If the natural numbers be uniformly distributed this reduces to 0.28. When the mantissas are uniformly distributed it becomes 0.27.

Reasoning as in Problem V, we may prove that when the natural numbers are restricted to 5 places, since 4.5 is the smallest tabular difference consistent with the condition that $c = c'$, the maximum error is $\frac{2.5}{4.5} + 0.50$, or 1.06. When the number is carried to 6 places, the maximum error is $10 . \frac{2.5}{4.5} + 0.5$ or 6.1.

I have taken 100 uniformly distributed 5 place mantissas and obtained their antilogarithms from Gernerth's table, carrying the results to 6 places and also to 5 places. More accurate values were then obtained from a 7-place table. Thus it was found that the average of the errors of the 6-place antilogarithms was 1.07 in units of the sixth place instead of 0.99 as the theory gives; the average of the errors of the 5-place antilogarithms was 0.27 in the fifth place, which accords well with the theory.

SOLUTIONS OF EXERCISES.

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THE curves of the family

$$\left(\frac{C}{r}\right)^n = \cos(p + n\vartheta).$$

where n is a parameter, all pass through a fixed point and cut orthogonally the fixed curve

$$\left(\frac{r}{c}\right)^n = \cos n\vartheta,$$

provided $C^n = \frac{1}{2}c^n \cos p$.

[Generalization of 14.] [*Alfred C. Lane*]

SOLUTION.

When $\vartheta = 0$ we have

$$r = C \,/\, \sqrt[n]{(\cos p)} = c \,/\, \sqrt[n]{2},$$

which is constant, since c and n are constants. All curves of the family pass, therefore, through the point whose polar co-ordinates are $c \,/\, \sqrt[n]{2}$, 0.